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On a class of multiplicative functions

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In a recent paper [1] we proved the following result: Let f be a multiplicative arithmetical function satisfying

$$\sum_{n \leqq x} |f(n)|^2 = O(x). \tag{1}$$

Then, for every irrational α, $(1/x) \sum_{n \leqq x} f(n) e^{2\pi i \alpha n}$ tends to zero as x tends to infinity.

We stated that this result still holds if the exponent 2 in (1) is replaced by any $\lambda > 1$ and that we planned to give the proof elsewhere. It turns out that this follows from the work of INDLEKOFER [3], for the hypothesis that

$$\sum_{n \leqq x} |f(n)|^\lambda = O(x) \quad \text{for some} \quad \lambda > 1 \tag{2}$$

implies that f belongs to his class $\mathcal{L}^*$. However we think it is still interesting to give our proof, which is quite different and enables us to obtain a uniform result, namely the following one.

Theorem. *Let K and λ be fixed real numbers satisfying $K \geqq 1$ and $\lambda > 1$, and let α be a fixed irrational number. Given any $\varepsilon > 0$, there exists a positive $X = X(K, \lambda, \alpha, \varepsilon)$ such that, for any multiplicative arithmetical function satisfying*

$$\sum_{n \leqq x} |f(n)|^\lambda \leqq Kx \quad \textit{for all positive } x, \tag{3}$$

we have

$$\left|(1/x) \sum_{n \leqq x} f(n) e^{2\pi i \alpha n}\right| \leqq \varepsilon \quad \textit{for} \quad x \geqq X.$$

For the proof it is obviously sufficient to consider the case when $\lambda \leqq 2$ for, by Hölder's inequality, (3) implies

$$\sum_{n \leqq x} |f(n)|^{\lambda'} \leqq K^{\lambda'/\lambda} x \quad \text{for any positive} \quad \lambda' < \lambda.$$

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Throughout this paper we write $e(t)$ for $e^{2\pi it}$; the letter p is used for a prime number, while the letters m and n are used to denote positive integers. $p^r\|n$ means "$p^r|n$ but $p^{r+1}\nmid n$". An empty sum is assumed to be zero.

1. To prove the theorem we need three lemmas.

1.1. Lemma 1. *Let K, M and ε be real numbers satisfying $M>1$ and $0<\varepsilon<M-1$. There exists $X(K, M, \varepsilon)>e$ such that, if g is any real-valued multiplicative function satisfying*

$$g(n)\geqq 0 \quad \textit{for every } n \textit{ and } \quad \sum_{n\leqq x} g(n)\leqq Kx \quad \textit{for every} \quad x\geqq 1,$$

then for $x\geqq X(K, M, \varepsilon)$

$$\sum_{\substack{p\leqq x\\ g(p)<M}} 1/p \geqq \big(1-(1+\varepsilon)/M\big)\log\log x.$$

Proof. As in the proof of Lemma 1 of [1] we have for $\sigma>1$ and $x>e$

$$\frac{1}{\log\log x}\sum_{\substack{p\leqq x\\ g(p)<M}}\frac{1}{p}\geqq\frac{1}{\log\log x}\sum_{p\leqq x}\frac{1}{p}-\frac{x^{\sigma-1}}{M\log\log x}\left(A+\log\frac{K\sigma}{\sigma-1}\right),$$

where $A=\sum\big(M/p-\log(1+M/p)\big)$.

Let η be a positive number such that $e^\eta<1+\varepsilon$. If $\sigma=1+\eta/\log x$, then, as x tends to infinity, the right-hand side of the above inequality tends to $1-e^\eta/M$. So there exists $X>e$ such that it is $\geqq 1-(1+\varepsilon)/M$ for $x\geqq X$.

1.2. Lemma 2. *Let $a_1, a_2, \dots, a_n, \dots$ be a sequence of complex numbers with the following properties:*

(i) *$|a_n|$ is a multiplicative function of n;*

(ii) *for some $\lambda\in]1, 2]$, $\sum_{n\leqq x}|a_n|^\lambda\leqq Kx$ for all positive x.*

Let $\mu=\lambda/(\lambda-1)$ (so that $1/\lambda+1/\mu=1$). If M is any positive number, then we have for every positive x

$$\sum_{\substack{p\leqq x\\ |a_p|\leqq M}}(1/p)\Big|(p/x)\sum_{\substack{n\leqq x\\ p\|n}}a_n-(1/x)\sum_{n\leqq x}a_n\Big|^\mu\leqq C_{\lambda,M}K^{1/(\lambda-1)},$$

where $C_{\lambda,M}$ is a constant which depends only upon λ and M.

Proof. It is known (Elliott [2]) that there exists an absolute contant C such that, if $x_1, x_2, \dots, x_n, \dots$ is any sequence of complex numbers, then for every positive x

$$\sum_{\substack{p,r\\ p^r\leqq x}}(1/p^r)\Big|(p^r/x)\sum_{\substack{n\leqq x\\ p^r\|n}}x_n-(1/x)\sum_{n\leqq x}x_n\Big|^2\leqq C(1/x)\sum_{n\leqq x}|x_n|^2,$$

and therefore

$$\sum_{p\le x}(1/p)\Big|(p/x)\sum_{\substack{n\le x\\ p\|n}}x_n-(1/x)\sum_{n\le x}x_n\Big|^2\le C(1/x)\sum_{n\le x}|x_n|^2.$$

We now consider a fixed positive x and we denote by $\mathscr{P}_x$ the set of those primes p which are $\le x$ and for which $|a_p|\le M$. The quantity to be estimated is

$$S=\sum_{p\in\mathscr{P}_x}(1/p)X_p^{\mu},\quad \text{where}\quad X_p=\Big|(p/x)\sum_{\substack{n\le x\\ p\|n}}a_n-(1/x)\sum_{n\le x}a_n\Big|.$$

We may suppose $S>0$ (and so $\mathscr{P}_x\ne\emptyset$) for the inequality to be proved is trivial if $S=0$. We may write $a_n=\alpha_n|a_n|$, where $|\alpha_n|=1$, and

$$(p/x)\sum_{\substack{n\le x\\ p\|n}}a_n-(1/x)\sum_{n\le x}a_n=\omega_pX_p,\quad \text{where}\quad |\omega_p|=1.$$

Define the function Φ on $\mathbf{C}$ by

$$\Phi(z)=(KS)^{-z}\sum_{p\in\mathscr{P}_x}(1/p)\bar{\omega}_pX_p^{\mu z}\Big((p/x)\sum_{\substack{n\le x\\ p\|n}}\alpha_n|a_n|^{\lambda z}-(1/x)\sum_{n\le x}\alpha_n|a_n|^{\lambda z}\Big).$$

(Here $0^u=0$ for any complex u). Φ is an entire function and it is bounded in every strip $A\le\operatorname{Re}z\le B$. We see that $\Phi(1/\lambda)=K^{-1/\lambda}S^{1/\mu}$ and that $|\Phi(z)|\le 1+M^{\lambda}$ when $\operatorname{Re}z=1$, for

$$\sum_{\substack{n\le x\\ p\|n}}|a_n|^{\lambda}=\sum_{\substack{mp\le x\\ p\nmid m}}|a_{mp}|^{\lambda}=|a_p|^{\lambda}\sum_{\substack{m\le x/p\\ p\nmid m}}|a_m|^{\lambda}\le|a_p|^{\lambda}\sum_{m\le x/p}|a_m|^{\lambda}.$$

Using Cauchy—Schwarz's inequality and Elliott's inequality, with $x_n=\alpha_n|a_n|^{\lambda z}$, we see that $|\Phi(z)|\le C^{1/2}$ when $\operatorname{Re}z=1/2$. It follows that, when $1/2\le\operatorname{Re}z\le 1$,

$$|\Phi(z)|\le C^{1-\operatorname{Re}z}(1+M^{\lambda})^{2(\operatorname{Re}z-1/2)}.$$

Taking $z=1/\lambda$ we get $K^{-1/\lambda}S^{1/\mu}\le C^{1/\mu}(1+M^{\lambda})^{2/\lambda-1}$, which yields

$$S\le C(1+M^{\lambda})^{(2-\lambda)/(\lambda-1)}K^{1/(\lambda-1)}.$$

1.3. Lemma 3. *Given an arithmetical function f and a real number α, set*

$$C_f(x,\alpha)=(1/x)\sum_{n\le x}f(n)e(n\alpha)\quad (x>0).$$

Now let $l_1, l_2, \dots, l_r$ be fixed positive numbers satisfying $l_j\ge l_1$ for $j>1$, and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be fixed real numbers satisfying $\alpha_j\not\equiv\alpha_k$ (mod 1) for $j\ne k$. Let

$$T=\sum_{j\ne k}1/|\sin\pi(\alpha_j-\alpha_k)|.$$

Then, for any arithmetical function f satisfying

(4) $$\sum_{n\le x}|f(n)|^{\lambda}\le Kx \quad \textit{for every positive } x \quad (K>0),$$

where $1<\lambda\leqq 2$, *we have for* $x\geqq T/\varepsilon$

$$\sum_{j=1}^{r}(1/l_j)|C_f(x/l_j,\alpha_j)|^\mu \leqq (1+\varepsilon)K^{1/(\lambda-1)}/l_1,$$

where $\mu=\lambda/(\lambda-1)$ (*so that* $1/\lambda+1/\mu=1$).

Proof. It is proved in [1] (Lemma 3) that there exists $X(\varepsilon)>0$ such that, for any arithmetical function f satisfying

$$\sum_{n\leqq x}|f(n)|^2\leqq Kx \quad \text{for every positive } x,$$

we have

$$\sum_{j=1}^{r}(1/l_j)|C_f(x/l_j,\alpha_j)|^2\leqq(1+\varepsilon)K/l_1 \quad \text{for} \quad x\geqq X(\varepsilon).$$

Although this is not stated explicitly, it is clear in the proof that $X(\varepsilon)$ may be taken equal to T/ε.

Now let f be any arithmetical function satisfying (4). Consider a fixed $x\geqq T/\varepsilon$ and set

$$|C_f(x/l_j,\alpha_j)|=Y_j,\quad C_f(x/l_j,\alpha_j)=Y_ju_j,\quad \text{where} \quad |u_j|=1,$$

$$f(n)=|f(n)|v_n,\quad \text{where} \quad |v_n|=1,$$

and

$$S=\sum_{j=1}^{r}Y_j^\mu/l_j.$$

We have to prove that $S\leqq(1+\varepsilon)K^{1/(\lambda-1)}/l_1$. Since this is trivially true if $S=0$, we may suppose $S>0$. Define the function Ψ on $\mathbf{C}$ by

$$\Psi(z)=l_1^{1-z}K^{-z}S^{-z}\sum_{j=1}^{r}\bar{u}_jY_j^{\mu z}\Big((1/x)\sum_{n\leqq x/l_j}|f(n)|^{\lambda z}v_ne(\alpha_jn)\Big).$$

Ψ is an entire function and it is bounded in every strip $A\leqq\operatorname{Re}z\leqq B$. We see that $\Psi(1/\lambda)=l_1^{1/\mu}K^{-1/\lambda}S^{1/\mu}$ and that $|\Psi(z)|\leqq 1$ for $\operatorname{Re}z=1$. If $z=1/2+iy$, where y is real, then by the Cauchy—Schwarz inequality

$$|\Psi(z)|\leqq l_1^{1/2}K^{-1/2}S^{-1/2}\Big(\sum_{j=1}^{r}Y_j^\mu/l_j\Big)^{1/2}\Big(\sum_{j=1}^{r}l_j|(1/x)\sum_{n\leqq x/l_j}|f(n)|^{\lambda(1/2+iy)}v_ne(\alpha_jn)|^2\Big)^{1/2},$$

that is

$$|\Psi(z)|\leqq l_1^{1/2}K^{-1/2}\Big(\sum_{j=1}^{r}(1/l_j)|C_{f_y}(x/l_j,\alpha_j)|^2\Big)^{1/2},\quad \text{where} \quad f_y(n)=|f(n)|^{\lambda(1/2+iy)}v_n.$$

Since $\sum_{n\leqq x}|f_y(n)|^2=\sum_{n\leqq x}|f(n)|^\lambda\leqq Kx$, it follows from the above quoted result that

$$\sum_{j=1}^{r}(1/l_j)|C_{f_y}(x/l_j,\alpha_j)|^2\leqq(1+\varepsilon)K/l_1.$$

We thus see that $|\Psi(z)|\leqq(1+\varepsilon)^{1/2}$ for Re $z=1/2$. It follows that $|\Psi(z)|\leqq(1+\varepsilon)^{1-\mathrm{Re}\,z}$ for $1/2\leqq\mathrm{Re}\,z\leqq 1$. In particular $|\Psi(1/\lambda)|\leqq(1+\varepsilon)^{1/\mu}$, or $l_1^{1/\mu}K^{-1/\lambda}S^{1/\mu}\leqq(1+\varepsilon)^{1/\mu}$, which yields the desired result.

2. Proof of the theorem. Let K and λ be fixed real numbers satisfying $K\geqq 1$ and $1<\lambda\leqq 2$, and let α be a fixed irrational number. Let $\mu=\lambda/(\lambda-1)$.

2.1. We first choose $M>1$ and η satisfying $0<\eta<M^\lambda-1$. By Lemma 1 we can choose $X_1>e$ such that, for any multiplicative function f satisfying (3),

$$\sum_{\substack{p\leqq x\\ |f(p)|<M}} 1/p \geqq \left(1-(1+\eta)/M^\lambda\right)\log\log x \quad \text{for} \quad x\geqq X_1. \tag{5}$$

2.2. Now we consider a fixed multiplicative function f satisfying (3) and we denote by $\mathscr{P}$ the set of those primes p for which $|f(p)|\leqq M$. By (5) $\mathscr{P}$ is infinite and its smallest element p_0 is $\leqq X_1$. We remark that, for each prime p,

$$\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)=\sum_{\substack{mp\leqq x\\ p\nmid m}} f(mp)e(\alpha mp)=f(p)\sum_{\substack{m\leqq x/p\\ p\nmid m}} f(m)e(p\alpha m)=$$
$$=f(p)\Big(\sum_{m\leqq x/p} f(m)e(p\alpha m)-\sum_{\substack{m\leqq x/p\\ p| m}} f(m)e(p\alpha m)\Big).$$

The first sum in the brackets is, in the notation of Lemma 3, $(x/p)C_f(x/p,\,p\alpha)$. On the other hand we have

$$\Big|\sum_{\substack{m\leqq x/p\\ p| m}} f(m)e(p\alpha m)\Big|\leqq\Big(\sum_{\substack{m\leqq x/p\\ p| m}}|f(m)|^\lambda\Big)^{1/\lambda}\Big(\sum_{\substack{m\leqq x/p\\ p| m}}1\Big)^{1/\mu}\leqq$$
$$\leqq (Kx/p)^{1/\lambda}(x/p^2)^{1/\mu}=K^{1/\lambda}x/p^{1+1/\mu}.$$

We thus see that

$$\Big|\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)\Big|\leqq|f(p)|(x/p)\big(|C_f(x/p,\,p\alpha)|+K^{1/\lambda}/p^{1/\mu}\big).$$

In particular, if $p\in\mathscr{P}$, then

$$\Big|p/x\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)\Big|\leqq M\big(|C_f(x/p,\,p\alpha)|+K^{1/\lambda}/p^{1/\mu}\big).$$

It follows that for every $p\in\mathscr{P}$

$$\Big|(1/x)\sum_{n\leqq x} f(n)e(\alpha n)\Big|\leqq$$
$$\leqq\Big|(p/x)\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)\Big|+\Big|(p/x)\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)-(1/x)\sum_{n\leqq x} f(n)e(\alpha n)\Big|\leqq$$
$$\leqq M|C_f(x/p,\,p\alpha)|+MK^{1/\lambda}/p^{1/\mu}+\Big|(p/x)\sum_{\substack{n\leqq x\\ p\| n}} f(n)e(\alpha n)-(1/x)\sum_{n\leqq x} f(n)e(\alpha n)\Big|$$

and therefore

$$\left|(1/x)\sum_{n\le x} f(n)e(\alpha n)\right|^\mu \le 3^{\mu-1}M^\mu |C_f(x/p,\,p\alpha)|^\mu + 3^{\mu-1}M^\mu K^{1/(\lambda-1)}/p +$$

$$+3^{\mu-1}\left|(p/x)\sum_{\substack{n\le x\\ p\|n}} f(n)e(\alpha n)-(1/x)\sum_{n\le x} f(n)e(\alpha n)\right|^\mu.$$

Using Lemma 2 with $a_n=f(n)e(\alpha n)$, we see that, if y is any number $\ge X_1$ (and therefore $\ge p_0$), then we have for $x\ge y$

$$\Big(\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p\Big)\left|(1/x)\sum_{n\le x} f(n)e(\alpha n)\right|^\mu \le$$

$$\le 3^{\mu-1}M^\mu \sum_{\substack{p\le y\\ p\in\mathcal{P}}}(1/p)|C_f(x/p,\,p\alpha)|^\mu + 3^{\mu-1}M^\mu K^{1/(\lambda-1)}\sum 1/p^2 + 3^{\mu-1}C_{\lambda,M}K^{1/(\lambda-1)}.$$

We now remark that, since α is irrational, if p' and p'' are distinct primes, then $p'\alpha\not\equiv p''\alpha \pmod 1$. Define a function T^* for $y\ge 3$ by

(6) $$T^*(y)=\sum_{\substack{p',\,p''\le y\\ p'\ne p''}} 1/|\sin \pi\alpha(p''-p')|.$$

It follows from Lemma 3 that, if $x\ge T^*(y)$, then

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}}(1/p)|C_f(x/p,\,p\alpha)|^\mu \le 2K^{1/(\lambda-1)}/p_0 \le K^{1/(\lambda-1)}.$$

Thus, if $y\ge X_1$ and $x\ge \operatorname{Max}(y,T^*(y))$, then

$$\Big(\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p\Big)\left|(1/x)\sum_{n\le x} f(n)e(\alpha n)\right|^\mu \le H,$$

where

(7) $$H=3^{\mu-1}K^{1/(\lambda-1)}\big(M^\mu+M^\mu\sum 1/p^2+C_{\lambda,M}\big).$$

Since

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p \ge \sum_{\substack{p\le y\\ |f(p)|<M}} 1/p \ge \big(1-(1+\eta)/M^\lambda\big)\log\log y,$$

this yields

(8) $$\left|(1/x)\sum_{n\le x} f(n)e(\alpha n)\right|^\mu \le H^{1/\mu}\big(1-(1+\eta)/M^\lambda\big)^{-1/\mu}(\log\log y)^{-1/\mu}.$$

2.3. So far, we have proved the following result: If $y\ge X_1$ and $x\ge \operatorname{Max}(y,T^*(y))$, where $T^*(y)$ is defined by (6), then for any multiplicative function f satisfying (3) we have (8), where H is defined by (7).

Given $\varepsilon>0$, we can choose $y_0\ge X_1$ such that the right-hand side of (8) is $\le\varepsilon^\mu$ for $y=y_0$. If $x\ge\operatorname{Max}(y_0,T^*(y_0))$, then for any multiplicative function f satisfying (3)

$$\left|(1/x)\sum_{n\le x} f(n)e(\alpha n)\right| \le \varepsilon.$$

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